Kuwait University | Math 101 | July 10, 2010
Department of Mathematics | First Exam | Time: 90 min.

Calculators and Mobile Phones are not allowed.

1. Evaluate each of the following limits, if it exists:

   a) $\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{x}}{x-2}$ (3 Points)

   b) $\lim_{x\to 0} \frac{\sin x}{\sqrt{x^3+2x^2}}$. (3 Points)

2. Use the definition of derivative to find $f'(x)$ if $f(x)=\sqrt{x+5}$. (3 Points)

3. Let $f(x)=x^4+x-10$. Use the Intermediate Value Theorem to show that the equation

$$f(x)=5$$

   has at least one real root. (3 Points)

4. Assume $\lim_{x\to 1} f(x)$ exists and

$$1 \le \frac{f(x)}{(x+1)^2} \le \frac{x^3-1}{3(x-1)}.$$

   Find $\lim_{x\to 1} f(x)$.

   (3 Points)

5. Let

$$f(x)=\frac{(x^2-4)\sqrt{x^2+6}}{(x^3+x^2-6x)}.$$

   a) Find and classify the discontinuities of f. (3 Points)

   b) Find the vertical and horizontal asymptotes for the graph of f, (if any).

   (3 Points)

6. Let $f(x)=\frac{x^{\frac{3}{5}}}{(2x+1)}$. Find the x-coordinate of the points on the graph of f at which:

   a) The tangent line is horizontal.

   b) The tangent line is vertical. (4 Points)

1. **a)** $\lim\limits_{x\to 2}\dfrac{\sqrt{4-x}-\sqrt{x}}{x-2}=\lim\limits_{x\to 2}\dfrac{\sqrt{4-x}-\sqrt{x}}{x-2}\times\dfrac{\sqrt{4-x}+\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}=\lim\limits_{x\to 2}\dfrac{-2(x-2)}{(x-2)(\sqrt{4-x}+\sqrt{x})}=\dfrac{-1}{\sqrt{2}}$

   b) $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x^3+2x^2}}=\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x^2(x+2)}}=\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x^2}\sqrt{x+2}}=\lim\limits_{x\to 0}\dfrac{\sin x}{|x|\sqrt{x+2}}$.

   $\lim\limits_{x\to 0^+}\dfrac{\sin x}{x}\times\lim\limits_{x\to 0^+}\dfrac{1}{\sqrt{x+2}}=1\times\dfrac{1}{\sqrt{2}}=\dfrac{1}{\sqrt{2}}$;

   $\lim\limits_{x\to 0^-}\dfrac{\sin x}{-x}\times\lim\limits_{x\to 0^-}\dfrac{1}{\sqrt{x+2}}=-1\times\dfrac{1}{\sqrt{2}}=-\dfrac{1}{\sqrt{2}}$. $\Longrightarrow\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x^3+2x^2}}$ Dose Not Exist.

2. $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{x+h+5}-\sqrt{x+5}}{h}=$
   $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h+5}-\sqrt{x+5}}{h}\times\dfrac{\sqrt{x+h+5}+\sqrt{x+5}}{\sqrt{x+h+5}+\sqrt{x+5}}=\lim\limits_{h\to 0}\dfrac{h}{h(\sqrt{x+h+5}+\sqrt{x+5})}=\dfrac{1}{2\sqrt{x+5}}$

3. Let $h(x)=f(x)-5\Rightarrow h(x)=x^4+x-15$. $h(0)=-5<0$, $h(2)=3>0\Rightarrow h(0)h(2)<0$. $h(x)$ is polynomial which is continuous on R, so it is continuous on $[0,2]$. Hence, by I.V.T. $\exists\ c\in(0,2)$ such that $h(c)=0\Rightarrow c^4+c-15=0\Rightarrow f(c)=5$

4. Using the Squeeze theorem: $1\le\frac{f(x)}{(x+1)^2}\le\frac{x^3-1}{3(x-1)}\Longrightarrow(x+1)^2\le f(x)\le\frac{(x+1)^2(x^3-1)}{3(x-1)}$.
   $\lim\limits_{x\to 1}(x+1)^2=4=\lim\limits_{x\to 1}\dfrac{(x+1)^2(x-1)(x^2+x+1)}{3(x-1)}=4\Longrightarrow\lim\limits_{x\to 1}f(x)=4$

5. **a)** The points of discontinuity are $x=0,2,-3$;
   At $x=0:\ \lim\limits_{x\to 0^{\pm}}f(x)=\pm\infty\Longrightarrow$ I.D. At $x=0$. $\Longrightarrow$ (V.A at $x=0$)
   At $x=-3:\ \lim\limits_{x\to -3^{\pm}}f(x)=\pm\infty\Longrightarrow$ I.D. at $x=-3$; $\Longrightarrow$ (V.A at $x=-3$)
   At $x=2:\ \lim\limits_{x\to 2^{\pm}}f(x)=\dfrac{2\sqrt{10}}{5}$ and $f(2)$ is undefined $\Longrightarrow$ R.D. at $x=2\Longrightarrow$ (No V.A at $x=2$)

   b) from part a) f has vertical asymptote at $x=0$ and $x=-3$.

   $f(x)=\frac{(x^2-4)\sqrt{x^2+6}}{(x^3+x^2-6x)}=\frac{(x^2-4)\sqrt{x^2(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}=\frac{(x^2-4)\sqrt{x^2}\sqrt{(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}=\frac{(x^2-4)|x|\sqrt{(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}$

   $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{(x^2-4)x\sqrt{(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}=\lim\limits_{x\to\infty}\dfrac{x^3(1-\frac{4}{x^2})\sqrt{(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}=1\Rightarrow y=1$ is H.A.
   $\lim\limits_{x\to-\infty}\dfrac{(x^2-4)(-x)\sqrt{(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}=\lim\limits_{x\to-\infty}\dfrac{-x^3(1-\frac{4}{x^2})\sqrt{(1+\frac{6}{x^2})}}{x^3(1+\frac{1}{x}-\frac{6}{x^2})}=-1\Rightarrow y=-1$ is H.A.

6. $f(x)=\frac{x^{\frac{3}{5}}}{(2x+1)}$. The domain of f is $R\backslash\{-\frac{1}{2}\}$.$f'(x)=\frac{\frac{3}{5}x^{-\frac{2}{5}}(2x+1)-2x^{\frac{3}{5}}}{(2x+1)^2}=\frac{-4x+3}{5x^{\frac{2}{5}}(2x+1)^2}$

   a) $f'(x)=0\Longrightarrow -4x+3=0\Longrightarrow$ H.T. at $x=\frac{3}{4}$.

   b) $f'(x)$ D.N.E. $\Longrightarrow x=0, x=-\frac{1}{2}$. At $x=-\frac{1}{2}\notin D_f=R\setminus\{-\frac{1}{2}\}\Longrightarrow$ No V.T. at $x=-\frac{1}{2}$. At $x=0$: f is continuous at $x=0$, $\lim\limits_{x\to 0}|f'(x)|=\infty\Longrightarrow$ V.T. at $x=0$